



TITLE:

Some martingale identities and inequalities(RECENT DEVELOPMENT IN MARTINGALE THEORY AND ITS APPLICATIONS)

AUTHOR(S):

Novikov, A.A.

CITATION:

Novikov, A.A.. Some martingale identities and inequalities(RECENT DEVELOPMENT IN MARTINGALE THEORY AND ITS APPLICATIONS). 数理解析研究所講究録 1983, 491: 107-113

ISSUE DATE:

1983-05

URL:

<http://hdl.handle.net/2433/103532>

RIGHT:

Some martingale identities and inequalities

A.A. Novikov

Steklov Math. Institute, Moscow

0. Summary. We consider moment type inequalities and identities for continuous time martingales. An application is given for processes with independent increments.

1. General denotations. We assume that (Ω, \mathcal{F}, P) is a complete probability space with a family (\mathcal{F}_t) , $t \geq 0$, of σ -algebras satisfying the standard conditions (nondecreasing, right continuous etc, [1], [2]; the paper [2] contains a short and good survey of the theory of stochastic integration). Any random process will be supposed to be \mathcal{F}_t -adapted and "cadlag" (that is right continuous and having limits on the left). Denote by M (M_{loc} , M_{loc}^C , M_{loc}^d , M_{loc}^2) classes of uniformly integrable martingales (local, continuous local, pure discontinuous local, local square-integrable, respectively) with respect to the family (\mathcal{F}_t) .

We denote by $p = p(\omega, dt, dx)$ the integral random measure of jumps $\Delta X_s = X_s - X_{s-}$ of some given (model) process X_s ,

$$p(\omega, (0, t], \Gamma) = \sum_{0 < s \leq t} I(\Delta X_s \in \Gamma), \Gamma \in B(R^m \setminus \{0\}).$$

Here $B(R^m \setminus \{0\})$ is a symbol of the Borel algebras of sets from $R^m \setminus \{0\}$.

Also, let $q = q(\omega, dt, dx)$ be a compensator, or the dual predictable projection to the measure p ([1], [2]). Stochastic integrals of predictable functions $f = f(\omega, s, x)$ with respect to the measures $q(\omega, ds, dx)$ and $p(\omega, ds, dx) - q(\omega, ds, dx)$ will be denoted by symbols

$$\int_0^t \int_{R^m \setminus \{0\}} f(\omega, s, x) \circ q(\omega, ds, dx)$$

and

$$\int_0^t \int_{R^m \setminus \{0\}} f(\omega, s, x) \circ (p(\omega, ds, dx) - q(\omega, ds, dx))$$

respectively. In what follows we omit the variables (ω, s, x) and index $R^m \setminus \{0\}$. These stochastic integrals are defined for proper classes of (vector-valued) functions f ([1], [2]). In particular,

$$\int_0^\infty \int \frac{|f|^2}{1+|f|^2} \circ q < \infty \text{ a.s.} \Rightarrow \int_0^t \int f \circ (p-q) \in M_{loc}^d.$$

If $\mu_t \in M_{loc}^2$ there exists a unique predictable process $\langle \mu \rangle_t$ such that $(|\mu_t|^2 - \langle \mu \rangle_t) \in M_{loc}$. Note that

$$\int_0^\infty \int |f|^2 \circ q < \infty \text{ a.s.} \Rightarrow \int_0^t \int f \circ (p-q) \in M_{loc}^2,$$

and moreover, if the compensator q is continuous that is

$$q(\omega, \{t\}, \Gamma) = 0 \text{ for any } \Gamma \in B(R^m \setminus \{0\}) \quad (1)$$

then

$$\langle \int_0^t \int f \circ (p-q) \rangle = \int_0^t \int |f|^2 \circ q.$$

The letter C will denote any positive constant. We shall use the usual denotations for a maximal functions,

$$\mu^* = \sup_{t \geq 0} |\mu_t|.$$

2. Structure of martingales under consideration. It is well-known that any process $\mu_t \in M_{loc}$ can be represented as a sum,

$$\mu_t = \mu_t^c + \mu_t^d, \quad (\mu_t^c \in M_{loc}^c, \mu_t^d \in M_{loc}^d).$$

In what follows we suppose that $\mu_0 = 0$ and

$$\mu_t^d = \int_0^t \int f \circ (p-q)$$

with a proper function f (predictable, $\int_0^t \int |f|^2 (1+|f|^2)^{-1} \circ q < \infty$ a.s.).

Note that if the model process X_t is a local martingale then ([1])

$$X_t = X_t^c + \int_0^t \int x \circ (p-q), \quad (X_t^c \in M_{loc}^c).$$

We suppose for simplicity of formulations that the compensator q is continuous. In this case we have a simple formula for the quadratic characteristic of a martingale $\mu_t \in M_{loc}^2$

$$\langle \mu \rangle_t = \langle \mu^c \rangle_t + \int_0^t \int |f|^2 \circ q.$$

3. Moment inequalities. In what follows we denote by $\phi(x)$, $x \geq 0$, a nondecreasing continuous function such that

$$\phi(2x) \leq C\phi(x) \quad \text{for all } x \geq 0 \quad \text{and} \quad \phi(0) = 0.$$

(F.e. $\phi(x) = x^p L(x)$, $0 \leq p < \infty$, with $L(x)$ being a slowly varying function).

Theorem 1. Let $\mu_t = \mu_t^c + \int_0^t \int f \circ (p-q)$, $\mu_t^c \in M_{loc}^c$, and the compensator q satisfy the condition (1). If ϕ is a concave function then for any $\alpha \in [1, 2]$

$$E\phi(\mu^{*\alpha}) \leq CE\phi(\langle \mu^c \rangle_\infty^{\alpha/2}) + CE\phi\left(\int_0^\infty \int |f|^\alpha \circ q\right)$$

where constants C depend only on ϕ and α .

If ϕ is a convex function then

$$CE\phi(\langle \mu \rangle_\infty^{1/2}) + CE\left(\int_0^\infty \int \phi(|f|) \circ q\right) \leq E\phi(\mu^*) \leq CE\phi(\langle \mu \rangle_\infty^{1/2}) + CE\left(\int_0^\infty \int \phi(|f|) \circ q\right),$$

where constants C depend only on ϕ .

Remarks. In case of $\phi(x) = x^p$ and an absolute continuous compensator q these inequalities were proved in [3]. In case of discrete time martingales similar inequalities can be found in [4], [5].

4. Moment identities. Here we consider one-dimensional martingales μ_t which have the representation mentioned above with a continuous compensator q .

Define polynomials $V_n(y_1) = V_n(y_1, \dots, y_n)$ by help of the next recurrent formulas

$$V_0(y_1) = 1, V_1(y_1) = 1,$$

$$V_{n+1}(y_1) = y_1 V_n(y_1) - \sum_{j=0}^{n-1} \binom{n}{j} y_{n+1-j} V_j(y_1), \quad (n=2, 3, \dots).$$

Note if $y_i = 0$ for $i \geq 3$ then

$$V_n(y_1) = y_2^{n/2} \text{He}_n\left(\frac{y_1}{\sqrt{y_2}}\right), \quad n = 1, 2, \dots,$$

where $\text{He}_n(x) = (-1)^n \exp(\frac{1}{2}x^2) \frac{d^n}{dx^n} \exp(-\frac{1}{2}x^2)$ are Hermitian polynomials.

We shall use the following denotations

$$\bar{V}_n(\mu_t) = V_n(\mu_t + C_1, \langle \mu \rangle_t + C_2, \int_0^t \int f^3 \circ q + C_3, \dots, \int_0^t \int f^n \circ q + C_n)$$

where C_i are some constants, and $\bar{V}_n(0) = V_n(C_1, C_2, \dots, C_n)$.

Theorem 2. Let $\mu_t = \mu_t^c + \int_0^t \int f \circ (p-q)$, $\mu_t^c \in M_{loc}^c$ and the compensator q satisfy the condition (1). If

$$E(\langle \mu \rangle_\infty^{1/2}) + E\left(\int_0^\infty \int |f|^{\alpha \circ q}\right)^{1/2} < \infty$$

for some $\alpha \in [1, 2]$, $n=1$, or

$$E(\langle \mu \rangle_\infty)^{n/2} + E\left(\int_0^\infty \int |f|^n \circ q\right) < \infty$$

for $n=2,3,\dots$, then

$$E\bar{V}_k(\mu_\infty) = \bar{V}_k(0) \quad (k=1,\dots,n). \quad (2)$$

Remarks. In case of an absolute continuous compensator q theorem 2 was proved in [3] (in a little different form).

The conditions of theorem 2 guarantee also that $V_n(\mu_t) \in M$ and moreover, that $E \sup_{t \geq 0} |V_n(\mu_t)| < \infty$.

5. An application. The moment identities (2) may be used, for example, for calculating moments of first passage time for processes with independent increments through moving boundaries. Some examples for the case when μ_t is a standard Wiener process can be found in [6], and in the recent paper Farebee [7]. Here is another example.

Let X_t be a stochastically continuous process with independent increments having only positive jumps (that is its spectral measure $Q(dx)$ equals zero for $x < 0$). Suppose $EX_1=0$, $EX_1^2=1$ (if it exists), and consider a stopping time

$$\sigma_a = \inf\{t \geq 0: X_t \leq at^{1/2}-b\}, \quad (a>0, b>0).$$

Then under the condition $E(X_1^+)^n < \infty$, $(n=1,2,\dots)$

$$E\sigma_a^{k/2} < \infty \iff a > \bar{z}_k, \quad k=1,2,\dots,n$$

where $\bar{z}_k = \max(z: H_k(z)=0)$.

The moments $E\sigma_a^{k/2}$ can be calculated by help of identities (2).

For example,

$$E\sigma_a^{1/2} = \frac{b}{a} \quad (a > 0), \quad E\sigma_a = \frac{b^2}{a^2-1} \quad (a > 1), \dots$$

6. The Wald identity for continuous martingale. In case of $\mu_t \in M_{loc}^C$ we can slightly weaken the condition of the theorem 2 (for $n = 1$).

Denote by N a class of nonnegative continuous nondecreasing functions such that

$$\int_1^\infty \frac{f(t)}{t^{3/2}} dt = \infty.$$

(F.e. $f(t) = \sqrt{t} (\log(t+1))^{-1} \in N$ and so on).

Theorem 3. Let $\mu_t \in M_{loc}^C$, $\langle \mu \rangle_\infty < \infty$ a.s. and $E|\mu_\infty| < \infty$.

If $f \in N$ then

$$Ef(\langle \mu \rangle_\infty) < \infty \implies E\mu_\infty = 0.$$

Remarks. If $P(\langle \mu \rangle_\infty > t)\sqrt{t} = o(1)$, $t \rightarrow \infty$, then there exists a function $f \in N$ such that $Ef(\langle \mu \rangle_\infty) < \infty$. This fact and the theorem [3] (in a little different form) were mentioned by the author in [8]. In the paper [9] it was shown that under the additional condition $\sup_{t \geq 0} E|\mu_t| < \infty$

$$\lim_{t \rightarrow \infty} P(\langle \mu \rangle_\infty > t)\sqrt{t} = 0 \iff \lim_{t \rightarrow \infty} E|\mu_\infty - \mu_t| = 0.$$

Our approach is based on some simple facts about first passage times for a wiener process and differs from [9] (see details of proofs in [10]).

References

- [1] J. Jacod, Calcul stochastique et problèmes de martingales, Lecture Notes in Mathematics 714, Springer-Verlag, New York (1979).
- [2] Yu. M. Kabanov, R. Sh. Liptser, and A. N. Shiriyayev, Absolute continuity and singularity of locally absolutely continuous probability distributions I., Math. USSR-Sbornik, 35, 5 (1979), pp.631-680.
- [3] A. A. Novikov, On discontinuous martingale, Theory Prob. Appl., XX, N1 (1975), 13-28.
- [4] D. L. Burkholder, Distribution function inequalities for martingales, Ann. Prob., 1, N1 (1973), 19-42.
- [5] Y. S. Chow, H. Teicher, Probability theory, N-Y, Springer-Verlag (1978).
- [6] A. A. Novikov, On stopping times for a wiener process, Theory Prob. Appl., XVI, N3 (1971), 458-465.
- [7] B. Farebee, Tests with parabolic boundary for the drift of a Wiener process, Ann. Stat., v.10, N3 (1982).
- [8] A. A. Novikov, A martingale approach to first passage problems and a new condition for the Wald's identity, Proc. of IFIP-WG 7/I Working Conference, Visegrad 1980, Lecture Notes in Control and Inform. Sci., v.36, I46-I56 .
- [9] J. Azema, R. Gundy, M. Yor, Sur L'intégrabilité uniforme des martingales continues, Lecture Notes in Math., 1980, v.784, 53-61.
- [10] A.A. Novikov, Martingale identities, inequalities and their applications to nonlinear boundary problems for random processes, Dissertation, Steklov Math. Institute, 1982.